A CHILD WANTS NICE AND NO MEAN NUMBERS

A child wants nice and no mean numbers

Mathematics in primary education

2nd edition

Major parts were written at the occasion of M's sixth birthday in 2012

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Supported by

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Wetenschappelijk bureau van het Sociaal Liberaal Forum
Scientific bureau of the Social Liberal Forum
The Hague

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2nd edition
ISBN 978-946367275-7
Published by MijnBestseller.nl

American Mathematical Society: MSC2010
00A35 Methodology and didactics
97-XX Mathematics education
97F02 Arithmetic, number theory ; Research exposition
97G70 Analytic geometry. Vector algebra
97H20 Elementary algebra

Journal of Economic Literature: JEL
P16 Political economy
I20 General education
D02 Institutions: Design, Formation, Operations, and Impact
O17 Formal and Informal Sectors • Shadow Economy • Institutional Arrangements

Nederlandstalige Uniforme Rubrieksindeling: NUR
841 Onderwijskunde
918 Wiskunde algemeen
780 Economie en bedrijf algemeen (Onderwijs van wiskunde en beheer ervan, Political Economy)

Prologue

Mathematics education is a mess. Earlier books *Elegance with Substance* (EWS) (2009, 2015) and *Conquest of the Plane* (COTP) (2011) [1] present a diagnosis:

Mathematicians are trained for abstraction while education is an empirical issue.

When abstract thinking mathematicians enter a classroom they meet with real live pupils. They must experience cognitive dissonance, and they tend to resolve this by relying on tradition. They will tend to regard themselves as evidence that this works. However, mathematical formats have grown historically. Those aren't necessarily designed for didactics. EWS and COTP re-engineer mathematics education for didactic purpose. Each nation is advised to have a parliamentarian enquiry into mathematics education, in order to identify the proper policy for improvement, and to make funds available for change.

This book looks at mathematics in primary education. Its contents can be included in the list of examples where tradition is not as friendly to pupils as can be re-engineered.

I am professionally involved in mathematics education at the level of highschool and the first year of higher education, and thus these thoughts on elementary school are prospective only. Perhaps the proper word is *amateurish.* My very plea is for professional standards, and thus I am sorry to say that I cannot provide this myself for elementary school. For example, Domahs et al. (eds) (2012) discuss finger counting and numerical cognition, with theory and empirical research: which I haven't read or studied, and thus it is quite silly of me to discuss the topic. This qualifier holds for this whole book.

My only defence for this book – or the articles that it collects and re-edits – is that I want to organise my thoughts on this. If parliaments will already need to investigate the issue, with much more funds than I can muster, then it seems acceptable that I organise my marginal comments on primary education too. There is also a good reason why I must collect my thoughts on this. Thinking about education in highschool and the first year of higher education caused questions about more elementary mathematics. It seems rather natural to wonder whether some issues cannot be dealt with in elementary school.

To be sure: it is not at all clear whether the world is served by this book. However, I am still under the impression that these articles support the general diagnosis in EWS and COTP. It may also be that my intuition is wrong and that the questions posed here have good answers, which I only missed because I did not study the issues fully. The book however achieves its goal when it provides some new ideas and perspectives for the *true* researchers of elementary education, and when it indeed provides some additional support for the general diagnosis of EWS and COTP that parliaments must take steps.

This book has a Dutch counterpart in Colignatus (2012a) that was written at the occasion of my son M.'s sixth birthday. These books only partly overlap. Various Dutch texts on local conditions are not interesting for an English translation. The present book includes some new articles since 2012. I thank Yvonne Killian for her permission to use some of her ideas on presenting the Pythagorean Theorem in elementary school.

A shocking discovery in 2014 w.r.t. Holland was that abstract thinking Hans Freudenthal (1905-1990) sabotaged the empirical theory by Pierre van Hiele (1909-2010); [2] see also the discussion in Colignatus (2014, 2015). Readers interested in primary education will not quickly read §15.2 of COTP on the right approach by Van Hiele and the erroneous approach by Freudenthal. For that reason page 135+ below copies that text.

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[1] Reviews by Gamboa (2011) and Gill (2012).

[2] https://boycottholland.wordpress.com/2014/07/06/hans-freudenthal-s-fraud/

Contents in brief

Contents

Introduction

The West reads and writes **text** from the left to the right, while Indian-Arabic **numbers** are from the right to the left. English pronounces 14 as *fourteen* instead of *ten & four,* and switches order from 21, to *twenty-one.* This order is already better, yet there still is an issue, for structurally the latter is *two·ten & one*. Pronunciation as *ten & four* and *two·ten & one* gives so much more clarity that pupils could learn arithmetic much faster. [3]

Seen from the perspective of the pupil, the traditional pronunciation can be called *mean* and the mathematically proper way is *nice*.

We can express this more diplomatically by referring to the place value system, a.k.a. the positional system. The numbers themselves already fully use the place value system. The traditional pronunciation only *partly* uses the place value system. The present suggestion is that the pronunciation of the numbers fully uses the place value system too.

In *ten* + *ten* = *two·ten,* the result is immediately available in the positional system itself. Thus it would also be advantageous for pupils to grow aware, as a learning goal by itself, not only of the positional system itself, since this is already a learning goal, but also of its relation to language.

Thus teaching arithmetic does not only deal with number but also language. Education errs in regarding English as perfect. English as a language appears to be a crummy dialect of mathematics. A new learning goal will be to recognise the dialect *for what it is*.

The key notion thus is to regard traditional English as a dialect indeed, and extend lessons on arithmetic with clarification of the dialect. This book develops the proposal (i) to teach in a nice language (ii) to clarify the translation of *nice* to *mean* so that pupils grow aware of the pitfalls in the dialect. The translation of mathematical pronunciation to standard English would be like handling any dialect. Given that children learn other languages with ease, while this concerns only a small set of words and concepts, this translation cannot be much of a burden. Perhaps the reluctance in the USA and the UK to learn other languages and accept dialects is a larger bottleneck than possible doubts about the didactic advantages of using mathematics.

The chapter *Marcus learns counting and arithmetic with ten* contains a stylized lesson for six-year olds. This is not intended for actual use in class but provides an example to start thinking about this for research and development. Six-year olds can still be orienting on left and right, for example (perhaps also because of this), and it could take more refined material to handle the issues (like particular feedback on progress and error).

Sadly, though, all fingers are already in use for the numbers 1 – 10, and there are no fingers available to practice on the decimal system itself. Perhaps lower arms help out.

The second type of issues below are more directly on arithmetic (algebra) and (analytic) geometry. The texts relate to the ideas of Pierre van Hiele and Dina Van Hiele - Geldof about levels of insight (understanding, abstraction). These didactic ideas directly transfer from my experience with highschool. Education in highschool requires algebraic insight, and this is based upon arithmetic mastered in elementary school. Van Hiele thought that algebra could be started in elementary school already, and would even be the best subject to start in elementary school with formal deduction and proof.

[3] The middle dots are unpronounced, and are better than hyphens in numbers, to prevent possible confusion with the negative sign.

Pierre van Hiele also proposed to have vectors in elementary school. He was hesitant about formal proofs with geometry there. Killian (2006) (2012) designed a proof of the Pythagorean Theorem that however feels very natural for this environment, and I have seen it work wonderfully for pupils in an enrichment course in elementary school.

This 2nd edition has a major update.

(1) There is a key role for the ampersand in the pronunciation of numbers that is now recognised – while this use of the ampersand was rejected in the first edition.
(2) I proposed Xur and Yur in 2008 [4], then $\Theta = 2\pi = 6.28...$ in 2011, and the name Archi for Θ in 2012. [5] Obviously π is handy in some formulas for which $\Theta\, 2^H$ might seem arcane (when unfamiliar). Thus I kept an eye open for new insights. It was a surprise to realise that pupils in elementary school may rather begin with disks and area before they proceed with circles and circumference. This became p131.
(3) Colignatus (2018a) is included here on p21. Research on number sense and competence in arithmetic tends to be invalid because of the issue on pronunciation. This research requires a standard on pronunciation in order to attain validity, even when education itself is slow in change. (An important step is that the German association Zwanzigeins, with chairman Peter Morfeld, takes an interest in this discussion. [6])
(4) Colignatus (2018b) further develops the relation between pronunciation and the education on the place value system. Different levels in the curriculum are recognised. This paper uses *Mathematica* and this software allows that one can actually hear how the numbers would be pronounced, and how things would be for kids who still live in a world of sounds (before reading and writing). This paper is not included here, and the reader is referred to the separate paper, also available in the Wolfram cloud.
(5) Colignatus (2018cd) on the negative numbers are mentioned in the section on arithmetic below, p89. The abstract of (2018c) is included on p102.
(6) Colignatus (2018e), here included on p145, is a letter to the makers of the US *Common Core State Standards* (CCSS) and *Trends in International Mathematics and Science Study* (TIMSS). This letter reports about the issue of this book, using this recent finding on the negative numbers as a hoped-for eye-opener.
(7) Within mathematics education research (MER) we must distinguish between traditional mathematics education (TME) and “reform” or “realistic” mathematics education (RME) and my own proposal of re-engineering mathematics education. [7] Unfortunately, there is a math war between TME and RME. [8] [9] A key factor is that mathematicians are trained on abstraction and not trained on empirical research. Testing on competence in mathematics is often delegated to psychometrics. However, psychometricians may lack understanding of didactics of mathematics. There is a letter to the integrity of research committee of Leiden University, p169, and a supporting analysis using institutional economics, p175. Remarkably this issue adds some 50 pages to this new edition, increasing the weight of the meta-argument.

Our order of discussion thus is: numbers, arithmetic, geometry, meta-commentary.

[4] http://www.wiskundebrief.nl/456.htm#2
[5] https://boycottholland.wordpress.com/2012/02/18/mathematical-constant-archimedes/
[6] https://zwanzigeins.jetzt/
[7] https://zenodo.org/communities/re-engineering-math-ed/about/
[8] https://boycottholland.wordpress.com/2016/01/24/graphical-displays-about-the-math-war/
[9] https://www.theglobeandmail.com/opinion/article-in-the-ongoing-math-wars-both-sides-have-a-point/

Medical School as a model for education

2014-07-18 [10]

In Medical School, doctors are trained while doing both research and treating patients. Theory and practice go hand in hand. We should have the same for education. Teachers should get their training while doing theory and learning to teach, without having to leave the building. When graduated, teachers might teach at plain schools, but keep in contact with their alma mater, and return on occasion for refresher updates.

Some speak about a new education crisis (e.g. in the USA). The above seems the best solution approach. It is also a model to reach all existing teachers who need retraining. Let us now look at the example of mathematics education.

Professor Hung-Hsi Wu [11] of UC at Berkeley is involved in improving K12 math education since the early 1990's. He explains how hard this is, see two enlightening short articles, one in the *AMS Notices* 2011 [12] and one interview in the *Mathematical Medley* 2012. [13] These articles are in fact remarkably short for what he has to tell. Wu started out rather naively, he confesses, but his education on education makes for a good read. It is amazing that one can be so busy for 30 years with so little success while around you Apple and Google develop into multi-billion dollar companies.

Always follow the money, in math education too. A key lesson is that much is determined by textbook publishers. Math teachers are held on a leash by the answers books that the publishers provide, as an episode of *The Simpsons* shows when Bart hijacks his teacher's answers book. [14] As a math teacher myself I tend to team up with my colleagues since some questions are such that you need the answers book to fathom what the question actually might be (and then rephrase it properly).

At one point, the publishers apparently even ask Wu whether he has an example textbook that they might use as a reference or standard that he wants to support. The situation in US math education appears to have become so bad that Wu discovers that he cannot point to any such book. Apparently he doesn't think about looking for a UK book or translating some from Germany or France or even Holland or Russia. In the interview, Wu explains that he only writes a teacher's education book now, and leaves it to the publishers to develop the derived books for students, with the different grade levels, teacher guides and answers books. One can imagine that this is a wise choice for what a single person can manage. It doesn't look like an encouraging situation for a nation of 317 million people. One can only hope that the publishers would indeed use quality judgement and would not be tempted to dumb things down to become acceptable to both teachers and students. In a world of free competition perhaps an English publisher would be willing to replace "rigour" by "rigor" and impose the A-levels also in the US of A.

In my book *Elegance with Substance* (2009, 2015) I advise the parliaments of democratic nations to investigate their national systems of education in mathematics. Reading the experience by Wu suggests that this still is a good advice, certainly for the US.

[10] https://boycottholland.wordpress.com/2014/07/18/the-medical-school-as-a-model-for-education/
[11] http://math.berkeley.edu/~wu/
[12] http://www.ams.org/notices/201103/rtx110300372p.pdf
[13] http://math.berkeley.edu/~wu/Interview-MM.pdf
[14] http://www.wired.com/2013/11/simpsons-math/

About the subject of logic, professor Wu in the interview p14 suggests that training math teachers in mathematical logic would not be so useful. He thinks that they better experience logic in a hands-on manner, doing actual proofs. I disagree. My book *A Logic of Exceptions* (1981 unpublished, 2007, 2011) would be quite accessible for math teachers, shows how important a grasp of formal logic is, and supports the teaching of math in fundamental manner. The distinction between necessary and sufficient conditions, for example, can be understood from doing proofs in geometry or algebra, but is grasped even better when the formal reasons for that distinction are seen. I can imagine that you want to skip some parts of ALOE but it depends upon the reader what parts those are. Some might be less interested in history and philosophy and others might be less interested in proof theory. Overall I feel that I can defend ALOE as a good composition, with some new critical results too.

Thus, apart from what parliaments do, I move that the world can use more logic, even in elementary school.

Update 2015:

Editing the 2nd edition of *Elegance with Substance* (2015), now available, I was struck again by the empirical observation on the diversity of students and pupils. Evidence based education (EBE) may never attain the sample sizes that are required for statistical testing of theories that allow for such diversity. This fits the Medical School model: there is an important role for individual observation and personal hands-on experience to deal with empirical variety. Methodology and statistics remain important, of course, but in balanced application.

It appears that professor Wu is updating some files. There is a rationale that such updates cause new file names and hence new links. A consequence is that old links break. My suggestion is to keep the old file names and links, and only insert the updated text. I have done so one my website and it works fine. Major changes can always be discussed in an appendix. Only fundamental new texts require new links.

One such update concerns professor Wu's text on fractions. [15] The text follows from professor Wu's objective to neatly develop the tradtional approach. Reading it again, I am struck again by the cumbersomeness of that approach. Much more elegant is the suggestion by Pierre van Hiele to abolish fractions, and use the multiplicative form. See this short introduction [16] and the longer discussion in *A child wants nice and no mean numbers* (2015).

Update 2018: On the latter, see p89 below.

[15] https://math.berkeley.edu/~wu/CCSS-Fractions_1.pdf (new link, as long as it lasts)
[16] https://boycottholland.wordpress.com/2014/09/04/with-your-undivided-attention/

English as a dialect for a didactic number system

The problem

The issue came to my attention by Gladwell (2008:228):

> "Ask an English-speaking seven-year-old to add *thirty-seven* plus *twenty-two* in her head, and she has to convert the words to numbers (37 + 22). Only then can she do the math: 2 plus 7 is 9 and 30 plus 20 is 50, which makes 59. Ask an Asian child to add *three·tens-seven* and *two·tens-two,* and then the necessary equation is right there, embedded in the sentence. No number translation is necessary: It's *five·tens-nine*." (Hyphen edited.)

My alternative suggestion is to use *five·ten & nine,* thus (i) no 'tens' and (ii) the use of a middle dot and ampersand (smaller font). The hyphen is unattractive since it is too similar to subtraction. The dot is not pronounced, like the hyphen or comma. Thus there is not only the notation of 59 and the pronunciation, but also the notation of the pronunciation. [17]

Gladwell (2008:228) also emphasizes the importance of mental working space:

> "(…) we store digits in a memory loop that runs for about two seconds."

English numbers are cumbersome to store. He quotes Stanislas Dehaene:

> "(…) the prize for efficacy goes to the Cantonese dialect of Chinese, whose brevity grants residents of Hong Kong a rocketing memory span of about 10 digits."

The problem has an internationally quick fix: Use the Cantonese system and sounds for numbers. It would be good evidence based education (EBE) to determine whether this would be feasible for an English speaking environment (e.g. start in Hong Kong).

Decimal system

There is more to it. The decimal number system has, for digits *a, b, c, d, ….*:

$$...dbca = a \times 10^0 + b \times 10^1 + c \times 10^2 + d \times 10^3 + \ldots$$

The West reads and writes text from the left to the right while Indian-Arabic numbers are from the right to the left. Thus 19 is *nineteen* instead of *ten & nine.* Human psychology apparently focuses on the lowest digits that have been learned first. The order switches in English at *twenty-one*, when attention shifts to the most important weight. While English switches order at 21, Dutch continues in the wrong order till 99 (*negen-en-negentig*). Thus instead of saying *...dcba* (most important weight first) we have reversed pronunciation ...*dcab* for the numbers below 20 (English) or 100 (Dutch). See Ejersbo & Misfeldt (2011) for the Danish convolutions.

Can we do something about these linguistic pecularities ? A key observation is that for higher numbers like 125 the Indian-Arabic writing order happily co-incides with our attention for the most important weights of the digits. Let this order be the guide. Let us agree that 21 is *two·ten & one.* The article *Marcus learns counting and arithmetic with ten* (page 35) explains how this works. The idea is that the most important weight is pronounced first, and that *ten* is the weight (and not *tens*).

Conclusion: We can apparently handle the pecularities of the natural languages. But also at an appalling cost of teaching in primary education. Instead, there is a number system with didactic clarity so that pupils could learn arithmetic more easily: the decimal

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[17] 2015: A relevant reference to Barrow (1993) is discussed on page 208 below.

positional system. The translation to English would be a mere matter of learning another dialect, which cannot be a burden given the ease by which children learn other languages, and also given the small set of words and concepts. Perhaps the English and American reluctance to learn other languages and accept dialects is a larger bottleneck than possible doubts about the didactic advantages. The key notion is to regard English as a dialect indeed, and extend lessons on arithmetic with clarification of the dialect.

Language pecularities

For English it may be easier to switch from *nineteen* to *ten & nine* and from *twenty* to *two·ten.* Other languages may have to make a greater adjustment. Consider Dutch as an example for handling such pecularities.

English distinguishes *ten* and *teen* in *nineteen* while Dutch uses *tien* everywhere, such as *negentien* for 19. A possible switch in Dutch to *tien & negen* runs against the problem that the new pronunciation of 90 would be *negen·tien* (English *nine·ten*). It would wreak havoc that the new pronunciation of 90 would be the old 19.

An option in Dutch is to use a new plural: *tienen* (rather than *tientallen* for the numbers of ten). However, the plural *tens* is not needed, and may cause later problems for higher powers such as *ten·ten·ten* for *thousand.* Thus *tens* and *tienen* can be used in discussion but not official pronunciation.

The solution in Dutch is to introduce a new label *tig* which can be done since 20 = *twintig* and 30 = *dertig* and so on. This is presented in Colignatus (2012a).

The equivalent for English would be to use *ty* [18] so that we would get *two·ty* and *three·ty*. The latter is not necessary since we can already use *ten.* Perhaps *two·ty* is better than *two·ten* but *ten* does fine. Better to have *hundred* = *ten·ten* than *ty·ty*.

English tends to use a hyphen: *twenty-two*. Dutch tends to concatenate words and has *tweeëntwintig* with the sudden umlaut to prevent "confusion" over vowels. (Thus an original confusion is solved by introducing another one.) For pupils learning the structure of the number system it is useful to avoid complexity. The middle dot then is better than a hyphen since the subject area is arithmetic and there might be a confusion with the minus sign. Thus Dutch *twee·tig & twee* is fine. Or switch to English or Cantonese.

Positional system and multiplication

It is a question at what age pupils can understand and actually learn multiplication. It is an option to see whether they already can multiply for the numbers up to 5 before progressing with the numbers above 20. When multiplication is known then it is easier to highlight the numerical structure. We can write (using 'times' and 'to time' rather than 'multi-plus' and 'multiplication'):

$$\ldots c \times \text{hundred} + b \times \text{ten} + a = \ldots cba$$

and then explain to the pupils, at least at some stage, that the number on the right is pronounced like on the left but without pronouncing "times" and pronouncing "plus" as "and". This is how the positional system supports understanding of arithmetic. At some points this may conflict with the assumed abstraction level of the pupils and perhaps the need to first learn to pronounce numbers before understanding the structure in the pronunciation. But when pupils are learning arithmetic, then we should also discuss how the positional system supports this.

[18] Ty has an origin like Gothic tigjus = tens, decades. https://www.etymonline.com/word/-ty

Notes on *Marcus learns counting and arithmetic with ten*

This discussion quickly becomes more complicated than needed. It is better to proceed with *Marcus learns counting and arithmetic with ten,* since this clarifies what the ideas entail. This is not spelling reform but targeted bilingualism. Please keep in mind:

(1) This text contains a stylized presentation for six-year olds. This is not intended for actual use in class but contains the framework for starting to think about that.
(2) The idea is to write *five·ten & nine,* where the dot is not pronounced and the order helps to decode the position.
(3) Much of arithmetic can be already done by proper pronunciation. Having this creates room for the operators plus and minus.
(4) Numbers are called *low* and *high* instead of *small* and *large,* since the latter would refer to absolute sizes, and a wrong convention might become a block in the later introduction of negative numbers.
(5) Putting the tables of addition together in a big table gives the opportunity to discuss patterns.
(6) Addition of many numbers uses the separation of numbers of ten (or higher) as an intermediate step. After some experience the pupils will use the direct method.

Multiplication is a long word

Before we can proceed with *Marcus* there is the issue that the word *multiplication* itself is long and rather awkward. In Dutch it is *vermenigvuldigen.* Apparently multiplication does not belong to the Indo-European core words like *mom* or *water.* Pupils in elementary school seem to lack easy words to express what they are doing.

Surprisingly, David Tall mentions that *of* is used for multiplication, see page 91 below. Thus *five of two makes ten* would be unambiguous.

I would explain that as *grouping five groups of two makes ten,* and then erase the *group* words.[19] We could call × the *of-sign*, and say *John ofs five and two to get ten,* rather than *John multiplies ....* The surface of a rectangle as *five by two makes ten* might perhaps also be used: *Mary bies two and five to get ten*. But verbs *to of* and *to by* are absent from online dictionaries and even Mathworld.[20]

My guess is that historically the development of *five of two is ten* into a verb *to of* was blocked by prim mathematicians who stuck to Latin *multiplex*. The Italian *volta* generates the English *times* with a reference to Father Time[21] – like in French *fois* and Dutch *keer, maal.*[22] Even when emphasis is put upon the notion of repetition, it is actually distractive. Multiplication is not only repetition of same sizes, but rather the *grouping* of those: creating a set of sets.

Times actually is a prefix *((five times) two) gives ten.* Five times two hamburgers need not be the same event as two times five hamburgers, if we allow for different days. The *times* prefix forces a demonstration that *((two times) five) gives ten too.* Once symmetry has been established we can create a *new* infix *five times two gives ten.* This is needlessly complex, and only gives an infix, i.e. without a rich and easy vocabulary with verb,

[19] Set theory has *joining five sets of two gives a set of ten.*
[20] http://mathworld.wolfram.com/Multiplication.html
[21] Amusing is http://math.stackexchange.com/questions/1150438/the-word-times-for-multiplication. But informative is Mauro Allegranza: "This latin plicō, like the ancient Greek : πλεκτός - "plaited, twisted", comes from Indo-European pleḱ- : “to plait, to weave”." Apparently related to *fold.* A weaving loom indeed reminds of a rectangle for multiplication. Folding a piece of paper however is an example for exponential growth.
[22] https://translate.google.com/#nl/en/keer

adjective and so on. It makes more sense to directly use an infix that actually has a rich and easy vocabulary that expresses symmetry directly.

The question becomes what synonyms for *times* there are. The Webster thesaurus on *times* is absent, with *time* only as a noun, and it is disappointing on *multiply.* [23] The idea of *double*, *triple*, *quadruple*, ... invites to think about a *multiple*, or *multi-plus,* indeed. But *run* is not *multi-walk.* When you are running then you don't want to be reminded continuously that you are actually walking but only faster.

It appears to be a relevant research objective to establish easy words for arithmetic so that pupils can discuss what they are doing without stumbling over the syllables and losing places in working memory. It is fine that mathematicians have developed the words *multiply* and *multiplication* so that adults may know what they are speaking about, but these words are overly complex for First or even Second Grade.

It is not clear how the verb *to group* is used for other applications, but if it is not used much then *group five of two makes ten* would be clearer than *times* on what multiplication is. My proposal in 2012 for Dutch was to use the verb *malen* (English *to mill*), given the already conventional Dutch *vijf maal twee geeft tien.* This was my first reaction to get rid of *vermenigvuldigen.* But *groepeer vijf van twee geeft tien* looks better.

For now, the paper *Marcus learns counting and arithmetic with ten* uses the verb *to time*. Hopefully there is scope for *to group*, or *to of* or *to by* eventually, with *tables of to of.*

Appendices

Some issues have been put in appendices.

Appendix: A novel way to look at numbers

An option is to mirror the numerals. Thus 19 becomes 91. It does not take much time to get used to, and when working from left to right then the handling of the overflow in addition feels rather natural, see Table 1.

Table 1. Addition, also in the mirror

4321	1234
765	567
98	89
0981	1890

However, the number system is well established, and given the psychological preference to know the size (the digit with the most weight) the present graphical order might be alright. A discussion on the four combinations of Indian / mirror and writing / pronouncing is put in **Appendix B**: *Number sense and sensical numbers.*

Appendix: Fingers and hand

Embodiment or gestures are important for the development of number sense. The decimal positional system can be supported by using fingers and ells (lower arms), see p83. See **Appendix C** on p215 on using fingers and hands. The particular system that is presented there will not be quickly used in first grade itself. It may be of use for students who are training for teachers in elementary school, and who want to re-experience what it is to learn a positional system.

[23] http://www.merriam-webster.com/thesaurus/multiply

The need for a standard for the mathematical pronunciation of the natural numbers. Suggested principles of design. Implementation for English, German, French, Dutch and Danish

September 2-9, 2015 & May 14 2018 (amendment on ampersand) & Sept 14 2018 [24]

Abstract

Current English for 14 is *fourteen* but mathematically it is *ten & four*. Research on number sense, counting, arithmetic and the predictive value for later mathematical abilities tends to be methodologically invalid when it doesn't measure true number sense that can develop when the numbers are pronounced in mathematical proper fashion. Researchers can correct by including proper names in the research design, but this involves some choices, and when each research design adopts a different scheme, also differently across languages, then results become incomparable. A standard would be useful, both ISO for general principles and national implementations. Research may not have the time to wait for such (inter-) national consensus. This article suggests principles of design and implementations for said languages. This can support the awareness about the need for a process towards ISO and national consensus, and in the mean time provides a baseline for research.

Keywords number sense, counting, arithmetic, mathematical ability, invalidity, design, standards, language, pronunciation, metastudy, number processing, numerical development, inversion effects, language-moderated effects, Google Translate

MeSH Terms Child, Child Development, Educational Measurement, Humans, Intelligence, Longitudinal Studies, Mathematics/education, Mathematics/methods, Mental Processes, Students

American Mathematical Society: MSC2010
- 00A35 Methodology and didactics
- 97F02 Arithmetic, number theory ; Research exposition

Journal of Economic Literature: JEL
- P16 Political economy
- I20 General education

Introduction

There is the distinction between (1) a mathematical pronunciation of the natural numbers (0, 1, 2, 3, ...) and (2) the pronunciation of those numbers in the natural languages (English, German, ..., French). While we will use the term "natural language" those languages clearly have been subjected to changes by influential authors and often even committees. Thus the present discussion on a standard on mathematical pronunciation is no breach upon nature itself.

Subsequently we observe that the distinction between (1) and (2) hinders research on number sense, counting and arithmetic, and their predictive value for later mathematical competence. Research methods may suffer from methodological invalidity when they mistake "number sense in natural language" for "true number sense with mathematical

[24] https://doi.org/10.5281/zenodo.774866

pronunciation". Researchers can try to correct by providing pupils with mathematical names, as Ejersbo & Misfeldt (2015) do. There is a risk that researchers implement their own interpretation of what mathematical names are, so that comparison of results becomes more and more difficult or impossible. Hence, a standard for such mathematical pronunciation will be useful, for achieving both validity and comparability.

For such a standard, we first establish the need, then propose principles of design, and then implement those principles to generate proposals for English, German, French, Dutch and Danish. It must be hoped that there will be a process towards consensus on such standards, both in ISO manner and national implementation. This article hopes to generate interest for such a process. In the mean time, researchers who are already in need of a baseline might be helped by the present suggestions.

The present issue differs principally from spelling reform. The spelling of a number ("29"), remains the same. Only its pronunciation changes. The new pronunciation will be spelled in common fashion too. This issue is not about spelling but about bilingualism and mathematical ability. A discussion in the media is by Shellenbarger (2014) in the WSJ.

The need for a standard

Professor Fred Schuh of TU Delft in 1943 observed that the Dutch pronunciation of the numbers was awkward. While English has *twenty-seven* in the order of written 27, Dutch has *zeven-en-twintig*. He again discussed this in Schuh (1949) and formulated a proposal for change, focussing on the numbers above 20. The proposal reached the Dutch minister of education, see Stoffels (1952), but it was not adopted.

Researchers in Norway had observed the same problem, and the Norse parliament (Storting) adopted a change in 1950, which we see reflected in the pronunciation after 1951. [25] I am not aware of an evaluation report. [26] Pixner et al. (2011) observe that the Czech language allows both kinds of pronunciation, and they show that the mathematical order causes less errors than the inverted order.

Various authors look into number sense, counting and arithmetic, in which there is an interplay of language, embodiment (fingers), nonsymbolic forms (e.g. dots), symbols (Indian - Arabic numbers), and working memory. Dowker & Roberts (2015) and Mark & Dowker (2015) compare English, Welsh and Cantonese. Zuber et al. (2009), Moeller et al (2011), Klein et al. (2013) indicate that inversion in German slows down the learning progress w.r.t. mathematics proper. In Holland, Friso - Van den Bos (2014), Xenidou-Dervou (2015) and Xenidou-Dervou et al. (2015) indicate the same for Dutch.

Hopefully this research generates interest amongst policy makers to adopt changes like in Norway 1950/51. However, such changes may still be limited w.r.t. a full mathematical pronunciation. Also English isn't perfect. It would be better to have ten & one for 11 and two·ten & one for 21. Thus the challenge is larger, also for English and Norse.

Recent studies that compare the performances in languages suffer from the problem that they may study the obvious. Schuh (1949) didn't need modern statistics to arrive at the logical conclusion that number-names are better pronounced as they are written. The real problem lies in the policy making process, see Colignatus (2015ab).

The research on the development of number sense tends to suffer from methodological invalidity. In truth, number sense is defined with the use of mathematical pronunciation. The reason for this is that numbers themselves are defined as such. A natural language tends to be a dialect of the mathematical pronunciation. One should not take a dialect as the norm. Studies that do not allow children to develop number sense by using the

[25] http://blogs.transparent.com/norwegian/learning-norwegian-numbers/
[26] I have asked this question at http://www.matematikksenteret.no/

mathematical names, will not observe true number sense, but "number sense in natural language". It may be admitted that one can develop statistical measures on such observations, but such a result is an awkward construct of both true number sense and confusion in language, in unclear mixture, without scientific relevance. [27]

The research on the development of number sense will also benefit from when researchers have deeper roots in mathematics education research (MER). The research quoted above derives mainly from the realm of (neuro-) psychology, and the problems on relevance, validity and comparability might have been observed at an earlier stage when there had been more awareness about what it actually is that pupils must learn. For a mathematician as Fred Schuh the pronunciation *zeven-en-twintig* is obviously illogical, while a neuro-psychologist may record it statistically as an "inversion", and actually think that this is how numbers are pronounced also mathematically, given that mathematicians also use such names. When (neuro-) psychologists would look deeper into MER, they must be warned that this field is not without problems of its own, however. See Colignatus (2015ab) for a longer discussion.

Relevant for research is the question whether pupils can deal with the difference between mathematical names and natural language dialect names. We see that many children can manage, see the examples of Czech, bilingual Chinese, bilingual English & X (e.g. in Holland), and in Ejersbo & Misfeldt (2015). The problem is not with children but in the policy making process, see Colignatus (2015ab).

Thus, researchers interested in number sense, validity and relevance, will tend to follow the example by Ejersbo & Misfeldt (2015) and include in the research design an instruction for pupils for using mathematical names. Perhaps researchers can find schools that are willing to participate in experiments with dual names, given that these aren't really much of experiments since we know that most children can deal with it. When parents are properly informed and first receive a training in the mathematical names, they might readily sign consent forms.

Colignatus (2015b) contains a chapter *Marcus learns counting and arithmetic with ten.* This text contains a stylized presentation for six-year olds. This is not intended for actual use in class but contains the framework for starting to think about that. There are translations for German, French, Danish and Dutch, that is: at this moment of writing the text still is in English but the numbers have been replaced by those in **Appendix D** below. This can also be used to instruct parents.

The real bottleneck then becomes comparability of research results. There are still questions of design. Different researchers might use different rules, and thus we would lose comparability. This establishes the need for a standard.

Principles of design

It is easy to suggest a "mathematical pronunciation of numbers in German", but what would that be ? When we use current *zehn* for 10, then there arises a problem, since the present pronunciation of 19 could be the mathematical pronunciation of 90. This will generate great confusion, and Germans would have to check continuously whether others are using current or mathematical names. However, German might replace *zehn* by *zig* or adopt English *ten* or scientific *deca* (though two syllables).

Number	*Math in English*	*English*	*Math in German ?*	*German*	*Math in German !*
19	ten & nine	nineteen	zehn & neun	neunzehn	**zig & neun**
90	nine·ten	ninety	neun·zehn	neunzig	**neun·zig**

[27] See also my weblog text https://boycottholland.wordpress.com/2015/08/29/research-on-number-sense-tends-to-be-invalid/

The proposed principles of design are:

(7) Pronunciation fully follows the place value system … c × hundred + b × ten + a = …*cba.* The current convention to start with the digit with the highest place value is fine. (See Colignatus (2015b) for lesser alternatives in pronunciation and order.) Much of arithmetic can be done by proper pronunciation (e.g. 2 × 10 + 4 = 24).

(8) In writing out the pronunciation, also in educational texts, the connectives middle-dot (unpronounced) and ampersand (pronounced) are used. We thus say *five·ten* & *nine* for 59, where the dot is not pronounced and the order helps to decode the position. The middle dot is preferred over the hyphen since the latter may be confused with the minus-sign. [28]

(9) **Insert August 20 2018:** (3a) For everyday use (in school) there is simplification in the pronunciation of 1 and 0. The proposed standard has simplified 11 = "ten & one" and not the nonsimplified "one·ten & one·one". (3b) On occasion the nonsimplified form can be used. A teaching objective is that pupils should understand the positional system, and the nonsimplified pronunciation indeed is more informative on this than the simplified pronunciation. However, while the nonsimplified form must be shown for such purpose, the everyday use is served by the simplified form. See Colignatus (2018b) for software that can show both forms, with default simplification. See below for more discussion of this aspect in education.

(10) There is awareness of the distinction between the process of calculation and the result given by the number. The process would be *two times ten plus four* and the result would be *two·ten* & *four.* On occasion *two of ten and four* might have the double role of both process and result. Operators might be bracketed or coloured it indicate that they are not pronounced, as in *two (times) ten plus four.* It must be tested whether young children would be served by a phase in which those operators are still pronounced also for the number result. Also elder pupils might at occasion be reminded of it. Also other names than times must be researched (e.g. the verb *to of*). Plus and minus however would be universal (given that "and" might not be commutative, as in *he missed the train and arrived late at work*).

(11) There are no exceptions in pronunciation of the digits in different place value positions. For example, German currently uses *sieben* in 7 and 27 and *sieb* in 70. A choice must be made for one name only. As a rule the shortest name is selected. For English some authors use *tens* as in *two·tens* & *one,* but *ten* is the value of the place, and must be used consistently. Multiplication can be scalar multiples (2 km) or consists of making groups, and can be expressed by the word *times,* or find another word that expresses this better, such as *grouping.*

(12) A key point for the standard is that it is identified where languages can make choices. Thus, a proposal for German identifies such a choice between *zig* and *ten.* It is up to German what it selects, but the standard helps German identify the choice.

(13) If the name of 10 cannot be used as a base (e.g. German *zehn* and Dutch *tien*) then it is tried to find a close substitute already in use (e.g. *zig* in German and *tig* in Dutch), while often a clear option is to use English *ten* or scientific *deca.*

(14) The above only gives the cardinals. There are also the ordinals (first, second, third, ...) and the fractions (that abuse the ordinals, e.g. "a fifth"). The fractions are solved by using $y\ x^H = y / x$ = "y per x" (H = -1). The ordinals are solved by adopting a single extension, e.g. English "th" (one-th, two-th, three-th, ...) or Dutch "de" (een-de, twee-de, drie-de, ...). There is no linguistic morphing (Dutch *tig-de* doesn't become *tig-ste*).

[28] See the use of the minus-sign in the place value system (a chapter in Colignatus (2015a)): https://boycottholland.wordpress.com/2014/08/30/taking-a-loss/

[29] Colloquial words like English *first* and French *premier* will gradually adopt a meaning of *"to begin with*" rather than an ordinal number.

(15) The rule is that mathematical names are used *in calculation*. The national natural language is explained as a dialect of mathematics. It is an explicit educational goal to identify the national language as such a dialect.

(16) It will be useful to denote mathematical pronunciation with a label, say *English-M* and *Deutsch-M*. This now holds for numbers but this may apply to more phenomena later on, notably for the vocabulary. This suits translations too, e.g. Google Translate.

(17) These principles are targeted at becoming a consensus ISO standard. Countries define their own mathematical pronunciation based upon such a standard, and include own national improvements. For example, 7 in Dutch is consistently *zeven* in 7, 27 and 70, but when Dutch changes, it might opt for a single syllable *zeef* anyway. English might prefer *thir* over *three,* with *thirteen, thirty* and *third* then becoming *ten* & *thir, thir·ten* and *thir-th.* (This choice though is not likely, because of potential confusion between *thir·ten* and *thirteen.*)

A suggestion is to have an expert meeting on this. In the mean time it still seems wise to provide this paper that identifies the issue. While the proposals in this paper may already be used in research to enhance comparability, ISO & national standards would be needed for further use such as in official education requirements (US Common Core) and eventually national adoption also in courts of justice.

Amendment May 14 2018

Colignatus (2018b) (update today or later) provides software in *Mathematica* to show how it all would hear and look, taking advantage of the modern facilities for sounds and translation. Revisiting the issue causes the following amendments.

(1) The symbol *Đ* (capital eth) can be used as symbolic 10, and be pronounced as "deka". The number 10 is universal already, but when each language pronounces it differently, then the universal pronunciation of *Đ* = 10 = deka may help at times. For example, $Đ^0$, $Đ^1$, $Đ^2$, $Đ^3$, ... indicates the place values and does not invite to do an actual calculation.

(2) It is better to use the (smaller) ampersand (&) to separate the place value positions. This is used above but is a major revision of the earlier text of 2015 and deserves clarification. Thus also for higher positions as e.g. 657 = six·hundred & five·ten & seven.

The connectives "&" and "·" have an important role in the pronunciation and writing of the words of the numbers. They differ from the mathematical operators "plus" and "group" (multi-plus), since + and × have commutation, association and distribution.

- The ampersand (&) is the ghost of addition, but simply "and", and not as the operator "plus" with all its properties. The ampersand should be pronounced to separate the place value positions. It is already (often) pronounced in German, Dutch and Danish, and other languages better adopt this practice too. It may take some time to get used to this but afterwards you will wonder why you never did before.
- The center dot (not pronounced) is the ghost of multiplication of the weight and the place value. It is not pure multiplication, like 5 days 2 hamburgers is not quite the same as 2 days 5 hamburgers.

Kids in kindergarten and Grade 1 live in a world of sounds. Thus it is important to also provide them with the &-separator of the place value positions, so that they have this anchor to distinguish which from what. For adults and native speakers of English it may

[29] See the importance of the ordinals for developing number sense (a chapter in Colignatus (2015a)): https://boycottholland.wordpress.com/2014/08/01/is-zero-an-ordinal-or-cardinal-number-q/