SUDOKU PATTERNS

To
Aad van de Wetering

Preface

For a long time I have fruitfully been co-operating with Aad van de Wetering in the area of sudoku. It all started end of 2005 when I managed to solve more or less logically one of his diabolical sudoku puzzles from Ukodus+ on his website home.planet.nl/~avdw3b/. He invited me to help him in writing a computer program that could solve in a human, i.e. logical, way sudoku puzzles instead of using brute force. My task was to specify the so-called *erasers*, algorithms that cancel numbers from the cells. In that connection the generic term of 'vlijn' (En. *fline*) was coined for any unit of nine cells that should contain the numbers 1 through 9. Unfortunately shortly after I was out of commission due to a personal incident.

The contact was renewed in March 2009. Under the continuous pressure of my requests Aad's program would quickly develop into a sophisticated machinery to design *exotic* sudokus. In the course of only three years he made over 800 fabulous puzzles. It was always a great pleasure to receive a puzzle from him, and still is, especially on the occasion of my birthday. Aad named his program *Xudoku*, the X referring to two crossing diagonals expressing his preference for the sudoku in which also the main diagonals contain the numbers 1 through 9. During the writing of this book I gratefully benefitted from his program.

Beginning of 2012 a selection of 160 puzzles was ready for publication under the title of *Exotische Sudoku's*. However it was hard to find a publisher until finally beginning of 2016 we discovered the self-publishing platform of *Brave New Books*, a more than great facility!

Having a mathematical background I was and am greatly interested in the combinatorics of the completed sudoku. How can sudokus be constructed? What patterns do they exhibit? What extra properties can be imposed on them? These extra properties or demands were often incorporated by Aad in his puzzles. His continuous attention to my ideas inspired me greatly. My theoretical investigations together with some of Aad's programming finds condensed into six articles in the 51st edition (2011–2012) of *Pythagoras*, a math journal for the youth.

The present book extends and deepens the ideas presented in the six articles (written in Dutch) and contains several new subjects. The motivation for writing this book in English is that some of the ideas are quite neat and deserve to be spread to a wider audience. At the same time this book is a *Call for Help* inviting readers to have fun in searching for more beautiful examples of sudokus and to do more research. It seems to me that I have barely scratched the many subjects in this book, none of them have been treated exhaustively. There is one subject that may be completed quite easily: settling the complete set of ML sudokus and their diverse properties, see Chapter 4. The most challenging task is posed in Chapter 2: trio-equivalence, a way of classifying sudokus via their trio structures. When you like combinatorics and permutations, programming in this area may be quite rewarding. I have only shown the tip of an iceberg.

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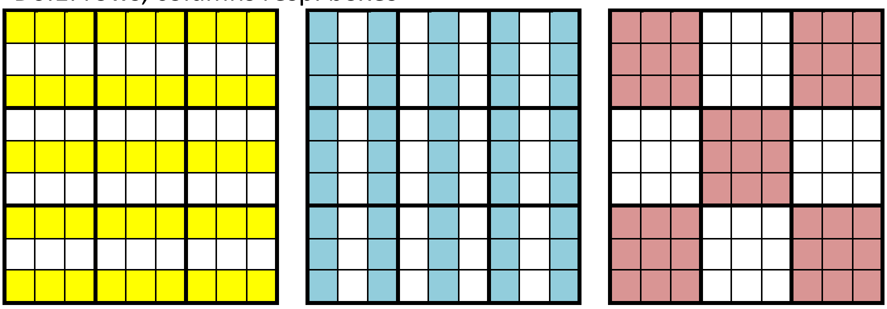
Definitions and Terminology

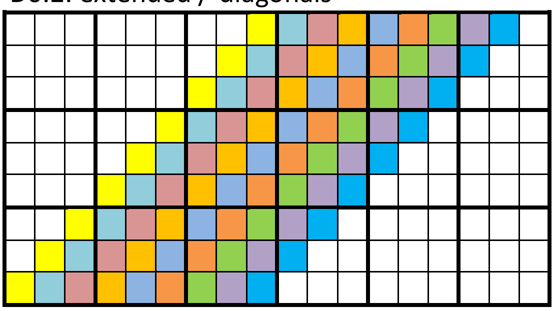
The standard 9×9 sudoku can in three ways be *partitioned*: into nine rows, nine columns and nine 3×3 boxes. The demand of the sudoku is that each of the rows, columns and boxes contains the numbers 1 through 9. A unit consisting of nine cells that should contain the numbers 1 through 9 is called here a *fline*, the word is a contraction of '*f*orm' and '*line*'. As a matter of fact the word is not completely a neologism, according to the *Urban Dictionary* 'fline' has the meaning of 'an attractive young female' (!) and is a combination of the words 'fly' with 'fine'. In short the flines of a sudoku are the rows, columns and boxes:

D0.1: rows, columns resp. boxes



The sudoku can also be partitioned into nine *extended* diagonals in the /-direction and nine *extended* diagonals in the \-direction. These are best viewed by sticking two copies of the sudoku together:

D0.2: extended /-diagonals



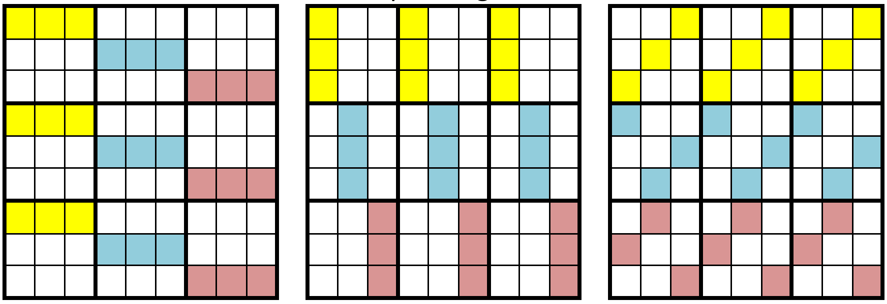
To get a completely coloured sudoku move the coloured triangle from the right to the left (or vice versa).

In the standard sudoku the (extended) diagonals are not flines. It is possible though that the diagonals in one direction are all flines, for that see Chapter 3.

A current extension of the sudoku contains two extra flines: the main diagonals, such a sudoku is called a *xudoku*, the ‘x’ mimics the two crossing diagonals.

In one case (see §1.7) it is demanded that the *broken* rows and *broken* columns are flines as well. The nine broken rows and nine broken columns are also partitions of the sudoku, moreover the nine broken /- or \-diagonals.

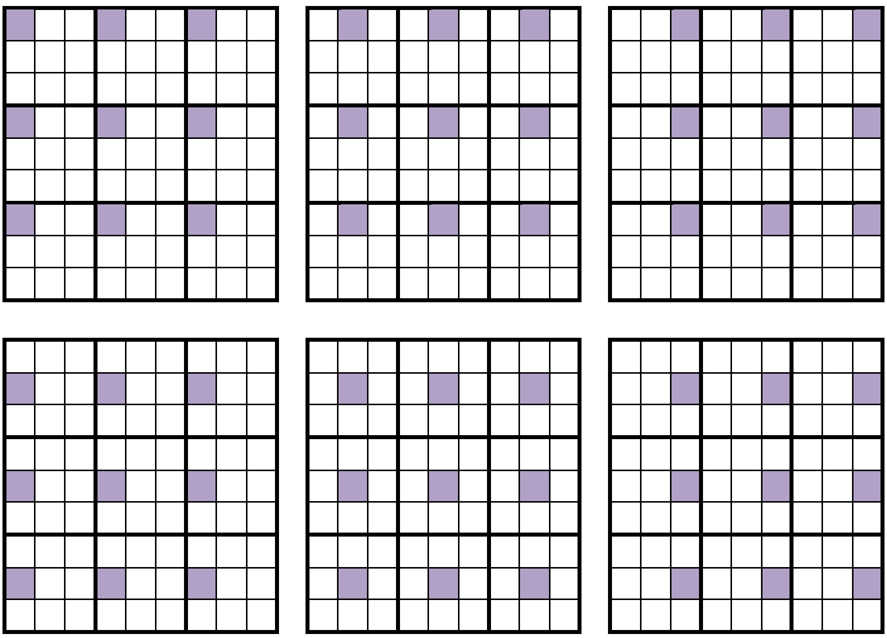
D0.3: broken rows, columns resp. /-diagonals

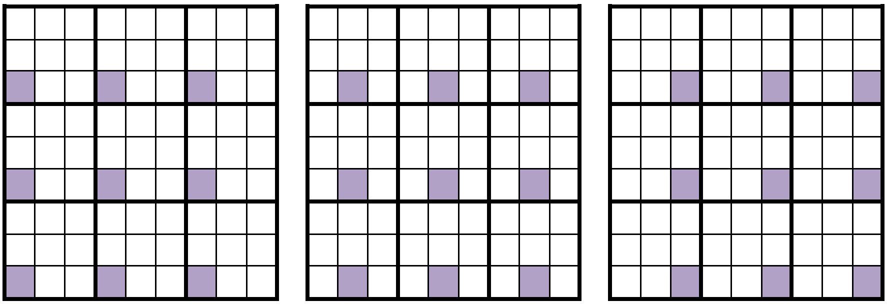


Check that D3.1 contains a partition of broken /-diagonals which are flines!

In one more, important, way the sudoku can be partitioned, into its nine *locations*. A location consists of the nine corresponding cells in the boxes. Especially in Chapter 4 the locations play a major role as flines; a sudoku in which all locations are flines is called *locative*.

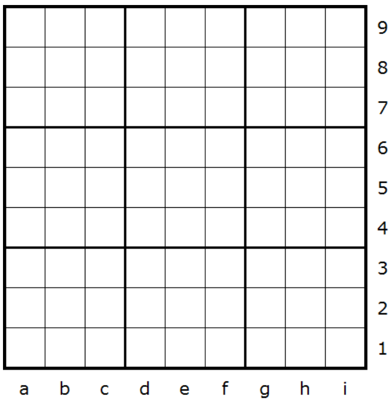
D0.4: the nine locations





To refer to the cells in the sudoku the chessboard coordinates are used: 1 through 9 for the rows, ‘a’ through ‘i’ for the columns:

D0.5: cell coordinates



Further a notation is needed to refer to the boxes, A through I:

D0.6: boxes

A	B	C
D	E	F
G	H	I

Three boxes next to each other are called a *band*, the horizontal bands are ABC, DEF and GHI, the vertical ones are ADG, BEH and CFI.

A *placement* is in a way the opposite of a fline, it consists of nine equal numbers instead of nine different ones. So a sudoku uniquely consists of nine placements. They play a major role in Chapter 4 where sudokus are *modularly* built from nine copies of one placement.

When none two of the equal numbers in a placement are in the same location that placement is called *locative*. The statement “a sudoku is locative” which means “its locations are flines” is equivalent with “its placements are locative”.

The term ‘modularly’ above still needs an explanation. The *modular* sudoku is obtained by identifying its left and right edges and its bottom and top edges. This has the advantage that the sudoku no longer has borders and placements can be moved on it at will, in Chapter 4 this is called *shifting*.

Another advantage is that extra conditions to the sudoku can be made uniform by demanding that they hold modularly. An example of that is the condition that any number has no *diagonal neighbours* equal to it. A sudoku obeying that condition may be called *tectonic*. An example follows, in D0.7 row 9 and column ‘a’ have been repeated to make it easier to check the condition modularly.

D0.7: a ‘tectonic’ sudoku

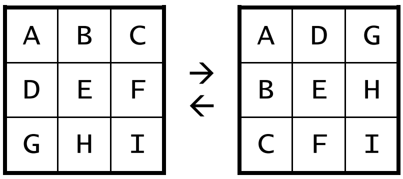
3	5	7	1	6	8	2	4	9	3
8	1	6	9	2	4	7	3	5	8
4	9	2	5	7	3	6	8	1	4
6	8	1	4	9	2	5	7	3	6
2	4	9	3	5	7	1	6	8	2
7	3	5	8	1	6	9	2	4	7
9	2	4	7	3	5	8	1	6	9
5	7	3	6	8	1	4	9	2	5
1	6	8	2	4	9	3	5	7	1
3	5	7	1	6	8	2	4	9	

The diagonal neighbours of e.g. c1 are b2, b9, d9 and d2, indeed these are not equal to 8. (NB: modularly the *orthogonal neighbours* of c1 are b1, c2, c9 and d1.) You also need to check that a9 ≠ i1 and a1 ≠ i9.

In the puzzle book *Exotische Sudoku’s* the condition of “diagonal neighbours are unequal”, as it was called there, plays (non-modularly though) a major role. By mistake in one of the puzzles this condition is absent; also in the puzzle on p. 159 this condition must hold. One other omission is that in the puzzle on p. 155 the numbers in the grey diagonals must be *different*. On the other hand in the present book the tectonic condition hardly plays a role.

An important notion is that of *equivalence*. Obviously when we apply a rotation or reflection (in an orthogonal or diagonal axis) to a sudoku, it remains a sudoku. (For these symmetries see §1.1.) Also when we interchange some bands or some rows/columns within a band. Further applying a permutation to the numbers keeps the sudoku correct. All the sudokus resulting from these operations – symmetries, interchanges and permutations – are considered the same, they are *equivalent*.

A counterexample to an action that yields no equivalence is switching the boxes B and D, C and G, F and H (it is different from the symmetry in the \-diagonal!):



for the simple reason that in general no correct sudoku results form that action. To get a correct sudoku it is necessary that the broken rows and broken columns are flines, as is for example the case in D1.11. However, in that special case the two sudokus can be shown to be equivalent. For that see the next page for the reason of keeping the diagrams together.

I suspect that in general given a sudoku with broken rows and broken columns as flines, the result by the above action is not equivalent to the initial sudoku. Can the reader maybe shed some light on that?

Throughout the book to any sudoku diagram the trio-information, *hvc*, of the sudoku is given, for details see §2.1. The *hvc* consists of three numbers:

h = # horizontal trios, v = # vertical trios, c = # common trios.

The *hvc* is an invariant of an equivalence class of sudokus. The main purpose of this book is to promote a simpler form of equivalence: trio-equivalence which is solely based on permutations of the extended trio-information of the bands.

An example of equivalence

D0.8

8	1	6	2	4	9	5	7	3
3	5	7	6	8	1	9	2	4
4	9	2	7	3	5	1	6	8
7	3	5	1	6	8	4	9	2
2	4	9	5	7	3	8	1	6
6	8	1	9	2	4	3	5	7
9	2	4	3	5	7	6	8	1
1	6	8	4	9	2	7	3	5
5	7	3	8	1	6	2	4	9

D0.9

8	1	6	7	3	5	9	2	4
3	5	7	2	4	9	1	6	8
4	9	2	6	8	1	5	7	3
2	4	9	1	6	8	3	5	7
6	8	1	5	7	3	4	9	2
7	3	5	9	2	4	8	1	6
5	7	3	4	9	2	6	8	1
9	2	4	8	1	6	7	3	5
1	6	8	3	5	7	2	4	9

D0.10

8	1	6	7	3	5	9	2	4
3	5	7	2	4	9	1	6	8
4	9	2	6	8	1	5	7	3
7	3	5	9	2	4	8	1	6
2	4	9	1	6	8	3	5	7
6	8	1	5	7	3	4	9	2
9	2	4	8	1	6	7	3	5
1	6	8	3	5	7	2	4	9
5	7	3	4	9	2	6	8	1

D0.9 is obtained from D0.8 by switching the boxes B and D, C and G, F and H. It is shown that D0.8 and D0.9 are equivalent.

In D0.8 first switch the vertical bands BEH and CFI, then appropriately permute columns in these bands such that D0.10 results. Next appropriately permute rows in the horizontal bands DEF and GHI such that D0.9 is returned.

As an exercise you can check that also rotating the boxes individually over 90° in D0.8 yields an equivalent sudoku.

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