CAMPUS HANDBOOK

BIEKE MASSELIS AND IVO DE PAUW

Multimedia Maths

LANNOO CAMPUS

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Content

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Chapter 1 · Arithmetic Refresher

As this chapter offers all necessary mathematical skills for a full mastering of all further topics explained in this book, we strongly recommend it. To serve its purpose, the successive paragraphs below refresh some required aspects of mathematical language as used on the applied level.

1.1 Algebra

Real Numbers

We typeset the set of:

- ▷ natural numbers (unsigned integers) as $\mathbb{N}$ including zero,
- ▷ integer numbers as $\mathbb{Z}$ including zero,
- ▷ rational numbers as $\mathbb{Q}$ including zero,
- ▷ real numbers (floats) as $\mathbb{R}$ including zero.

All the above make a chain of subsets: $\mathbb{N} \subset \mathbb{Z} \subset \mathbb{Q} \subset \mathbb{R}$.

To avoid possible confusion, we outline a brief glossary of mathematical terms. We recall that using the correct mathematical terms reflects a correct mathematical thinking. Putting down ideas in the correct words is of major importance for a profound insight.

Sets

- ▷ We recall writing all **subsets** in between braces, e.g. the **empty set** appears as $\{\}$.
- ▷ We define a **singleton** as any subset containing only one element, e.g. $\{5\} \subset \mathbb{N}$, as a subset of natural numbers.
- ▷ We define a **pair** as any subset containing just two elements, e.g. $\{115, -4\} \subset \mathbb{Z}$, as a subset of integers. In programming the boolean values *true* and *false* make up a pair $\{true, false\}$ called the boolean set which we typeset as $\mathbb{B}$.
- ▷ We define $\mathbb{Z}^- = \{\ldots, -3, -2, -1\}$ whenever we need negative integers only. We express symbolically that -1234 is an **element** of $\mathbb{Z}^-$ by typesetting $-1234 \in \mathbb{Z}^-$.
- ▷ We typeset the **setminus** operator to delete elements from a set by using a backslash, e.g. $\mathbb{N} \setminus \{0\}$ reading all natural numbers except zero, $\mathbb{Q} \setminus \mathbb{Z}$ meaning all pure rational numbers after all integer values left out and $\mathbb{R} \setminus \{0, 1\}$ expressing all real numbers apart from zero and one.

Calculation basics

operation	example	a	b	c
to add	$a+b=c$	term	term	sum
to subtract	$a-b=c$	term	term	difference
to multiply	$a \cdot b=c$	factor	factor	product
to divide	$\frac{a}{b}=c, b \neq 0$	numerator	divisor or denominator	quotient or fraction
to exponentiate	$a^b=c$	base	exponent	power
to take root	$\sqrt[b]{a}=c$	radicand	index	radical

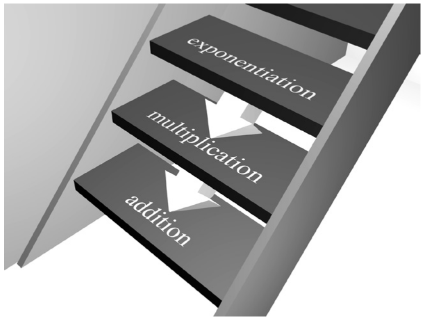
We write the **opposite** of a real number r as $-r$, defined by the sum $r+(-r)=0$. We typeset the **reciprocal** of a nonzero real number r as $\frac{1}{r}$ or r^{-1}, defined by the product $r \cdot r^{-1}=1$.

We define **subtraction** as equivalent to adding the opposite: $a-b=a+(-b)$. We define **division** as equivalent to multiplying with the reciprocal: $a:b=a \cdot b^{-1}$.

When we mix operations we need to apply priority rules for them. There is a fixed priority list 'PEMDAS' in performing mixed operations in $\mathbb{R}$ that can easily be memorized by 'Please Excuse My Dear Aunt Sally'.

- ▷ First process all that is delimited in between Parentheses,
- ▷ then Exponentiate,
- ▷ then Multiply and Divide from left to right,
- ▷ finally Add and Subtract from left to right.

Now we discuss the **distributive law** ruling within $\mathbb{R}$, which we define as threading a 'superior' operation over an 'inferior' operation. Conclusively, distributing requires two *different* operations.



Hence we distribute *exponentiating* over *multiplication* as in $(a \cdot b)^3 = a^3 \cdot b^3$. Likewise rules *multiplying* over *addition* as in $3 \cdot (a+b) = 3 \cdot a + 3 \cdot b$.

However we should never stumble on this 'Staircase of Distributivity' by stepping it too fast:

$$(a+b)^3 \neq a^3 + b^3,$$

$$\sqrt{a+b} \neq \sqrt{a} + \sqrt{b},$$

$$\sqrt{x^2+y^2} \neq x+y.$$

Fractions

A **fraction** is what we call any rational number written as $\frac{t}{n}$ given $t, n \in \mathbb{Z}$ and $n \neq 0$, wherein t is called the **numerator** and n the **denominator**. We define the reciprocal of a nonzero fraction $\frac{t}{n}$ as $\frac{1}{\frac{t}{n}} = \frac{n}{t}$ or as the power $\left(\frac{t}{n}\right)^{-1}$. We define the opposite fraction as $-\frac{t}{n} = \frac{-t}{n} = \frac{t}{-n}$. We summarize fractional arithmetics:

sum	$\frac{t}{n} + \frac{a}{b} = \frac{t \cdot b + n \cdot a}{n \cdot b}$,
difference	$\frac{t}{n} - \frac{a}{b} = \frac{t \cdot b - n \cdot a}{n \cdot b}$,
product	$\frac{t}{n} \cdot \frac{a}{b} = \frac{t \cdot a}{n \cdot b}$,
division	$\frac{\frac{t}{n}}{\frac{a}{b}} = \frac{t}{n} \cdot \frac{b}{a}$,
exponentiation	$\left(\frac{t}{n}\right)^m = \frac{t^m}{n^m}$,
singular fractions	$\frac{1}{0} = \pm\infty$ infinity,
	$\frac{0}{0} =?$ indeterminate.

Powers

We define a **power** as any real number written as g^m , wherein g is called its **base** and m its **exponent**. The opposite of g^m is simply $-g^m$. The reciprocal of g^m is $\frac{1}{g^m} = g^{-m}$, given $g \neq 0$.

According to the exponent type we distinguish between:

$$
\begin{aligned}
&g^3 = g \cdot g \cdot g && 3 \in \mathbb{N},\\
&g^{-3} = \tfrac{1}{g^3} = \tfrac{1}{g \cdot g \cdot g} && -3 \in \mathbb{Z},\\
&g^{\frac{1}{3}} = \sqrt[3]{g} = w \Leftrightarrow w^3 = g && \tfrac{1}{3} \in \mathbb{Q},\\
&g^0 = 1 && g \neq 0.
\end{aligned}
$$

Whilst calculating powers we may have to:

$$
\begin{aligned}
&\text{multiply} && g^3 \cdot g^2 = g^{3+2} = g^5,\\
&\text{divide} && \tfrac{g^3}{g^2} = g^3 \cdot g^{-2} = g^{3-2} = g^1,\\
&\text{exponentiate} && \left(g^3\right)^2 = g^{3 \cdot 2} = g^6 \ \text{ them.}
\end{aligned}
$$

We insist on avoiding typesetting radicals like $\sqrt[7]{g^3}$ and strongly recommend their contemporary notation using radicand g and exponent $\frac{3}{7}$, consequently exponentiating g to $g^{\frac{3}{7}}$. We recall the fact that all square roots are non-negative numbers, $\sqrt{a} = a^{\frac{1}{2}} \in \mathbb{R}^+$ for $a \in \mathbb{R}^+$.

As well knowing the above exponent types as understanding the above rules to calculate them are inevitable to use powers successfully. We advise memorizing the integer squares running from $1^2 = 1$, $2^2 = 4$, ..., up to $15^2 = 225$, $16^2 = 256$ and the integer cubes running from $1^3 = 1$, $2^3 = 8$, ..., up to $7^3 = 343$, $8^3 = 512$ in order to easily recognize them.

Recall that the only way out of any power is exponentiating with its reciprocal exponent. For this purpose we need to exponentiate both left hand side and right hand side of any given relation (see also paragraph 1.2).

Example: Find x when $\sqrt[7]{x^3} = 5$ by exponentiating this power.

$$x^{\frac{3}{7}} = 5 \Longleftrightarrow \left(x^{\frac{3}{7}}\right)^{\frac{7}{3}} = (5)^{\frac{7}{3}} \Longleftrightarrow x \approx 42.7494.$$

We emphasize the above strategy as the only successful one to free base x from its exponent, yielding its correct expression numerically approximated if we like to.

Example: Find x when $x^2 = 5$ by exponentiating this power.

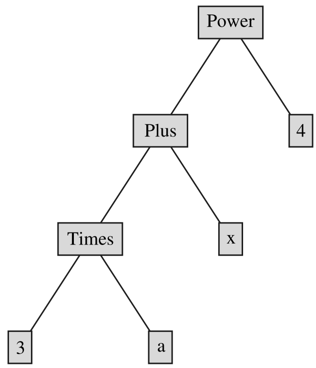
$$x^2 = 5 \Longleftrightarrow \left(x^2\right)^{\frac{1}{2}} = (5)^{\frac{1}{2}} \text{ or } -(5)^{\frac{1}{2}} \Longleftrightarrow x \approx 2.23607 \text{ or} -2.23607.$$

We recall the above double solution whenever we free base x from an *even* exponent, yielding their correct expression as accurate as we like to.

Mathematical expressions

Composed mathematical expressions can often seem intimidating or cause confusion. To gain transparancy in them, we firstly recall indexed variables which we define as subscripted to count them: $x_1, x_2, x_3, x_4, \dots, x_{99999}, x_{100000}, \dots$, and $\alpha_0, \alpha_1, \alpha_2, \alpha_3, \alpha_4, \dots$. It is common practice in industrial research to use thousands of variables, so just picking unindexed characters would be insufficient. Taking our own alphabet as an example, it would only provide us with 26 characters.

We define finite expressions as composed of (mathematical) operations on objects (numbers, variables or structures). We can for instance analyze the expression $(3a+x)^4$ by drawing its **tree form**. This example reveals a Power having exponent 4 and a subexpression in its base. The base itself yields a sum of the variable x Plus another subexpression. This final subexpression shows the product 3 Times a.



Let us also evaluate this expression $(3a+x)^4$. Say $a = 1$, then we see our expression partly collaps to $(3+x)^4$. If we on top of this assign $x = 2$, our expression then finally turns to the numerical value $(3+2)^4 = 5^4 = 625$.

When we expand this power to its **pure sum expression** $81a^4 + 108a^3x + 54a^2x^2 + 12ax^3 + x^4$, we did nothing but *reshape* its **pure product expression** $(3a+x)^4$.

We warn that trying to solve this expression - which is not a relation - is completely in vain. Recall that inequalities, equations and systems of equations or inequalities are the only objects in the universe we can (try to) solve mathematically.

Relational operators

We also refresh the use of correct terms for inequalities and equations.

We define an **inequality** as any *variable* expression comparing a left hand side to a right hand side by applying the 'is-(strictly)-less-than' or by applying the 'is-(strictly)-greater-than' operator. For example, we can read $(3a+x)^4 \leqslant (b+4)(x+3)$ containing variables *a, x, b*. Consequently we may solve such inequality for any of the unknown quantities a, x or b.

We define an **equation** as any *variable* expression comparing a left hand side to a right hand side by applying the 'is-equal-to' operator. For example $(3a+x)^4 = (b+4)(x+3)$ is an equation containing variables *a, x, b*. Consequently we also may solve equations for any of the unknown quantities a, x or b.

We define an **equality** as a constant relational expression being *true*, e.g. $7 = 7$. We define a **contradiction** as a constant relational expression being *false*, e.g. $-10 > 5$.

Real Polynomials

We elaborate upon the mathematical environment of polynomials over the real numbers in their variable or indeterminate x, a set we denote with $\mathbb{R}_{[x]}$.

▷ Monomials

We define a **monomial** in x as any product ax^n, given $a \in \mathbb{R}$ and $n \in \mathbb{N}$. We can extend this concept to several indeterminates $x, y, z, \ldots$ like the monomials $3(xy)^6$ and $3(x^2y^3z^6)$ are.

We define the **degree** of a monomial ax^n as its natural exponent $n \in \mathbb{N}$ to the **indeterminate part** x. We say constant numbers are monomials of degree 0 and linear terms are monomials of degree 1. We say squares to have degree 2 and cubes to have degree 3, followed by monomials of higher degree.

For instance the real monomial $-\sqrt{12}x^6$ is of degree 6. Extending this concept, the monomial $3(xy)^6$ is of degree 6 in xy and the monomial $3(x^2y^3z^6)^9$ is of degree 9 in $x^2y^3z^6$.

We define **monomials of the same kind** as those having an identical indeterminate part. For instance both $\frac{5}{7}x^6$ and $-\sqrt{12}x^6$ are of the same kind. Extending the concept, likewise $\frac{5}{7}x^3y^5z^2$ and $-\sqrt{12}x^3y^5z^2$ are of the same kind.

All basic operations on monomials emerge simply from applying the calculation rules of fractions and powers.

▷ Polynomials

We define a **polynomial** $V(x)$ as any sum of monomials. We define the **degree** of $V(x)$ as the maximal exponent $m \in \mathbb{N}$ to the indeterminate variable x. For instance the real polynomial

$$V(x) = 17x^2 + \frac{1}{4}x^3 + 6x - 7x^2 - \sqrt{12}x^6 - 13x - 1,$$

is of degree 6.

Whenever monomials of the same kind appear in it, we can simplify the polynomial. For instance our polynomial simplifies to $V(x) = 10x^2 + \frac{1}{4}x^3 - 7x - \sqrt{12}x^6 - 1$.

Moreover, we can sort any given polynomial either in an ascending or descending way according to its powers in x. Sorting our polynomial $V(x)$ in an ascending way yields $V(x) = -1 - 7x + 10x^2 + \frac{1}{4}x^3 - \sqrt{12}x^6$. Sorting $V(x)$ in a descending way yields $V(x) = -\sqrt{12}x^6 + \frac{1}{4}x^3 + 10x^2 - 7x - 1$.

Eventually we are able to evaluate any polynomial, getting a numerical value from it. For instance evaluating $V(x)$ in $x = -1$, yields $V(-1) = -\sqrt{12}(-1)^6 + \frac{1}{4}(-1)^3 + 10(-1)^2 - 7(-1) - 1 = -\sqrt{12} - \frac{1}{4} + 16 = \frac{63}{4} - 2\sqrt{3} \in \mathbb{R}$.

▷ Basic operations

Adding two monomials of the same kind: we add their coefficients and keep their indeterminate part

$$5a^2 - 3a^2 = (5-3)a^2 = 2a^2.$$

Multiplying two monomials of any kind: we multiply both their coefficients and their indeterminate parts

$$-5ab \cdot \frac{7}{4}a^2b^3 = -5 \cdot \frac{7}{4} \cdot a^{1+2}b^{1+3} = \frac{-35}{4}a^3b^4.$$

Dividing two monomials: we divide both their coefficients and their indeterminate parts

$$\frac{-8a^6b^4}{-4a^4} = \frac{-8}{-4}\,a^{6-4}b^{4-0} = 2a^2b^4.$$

Exponentiating a monomial: we exponentiate each and every factor in the monomial

$$\left(-2a^2b^4\right)^3 = (-2)^3(a^2)^3(b^4)^3 = -8a^6b^{12}.$$

Adding or subtracting polynomials: we add or subtract all monomials of the same kind

$$(x^2 - 4x + 8) - (2x^2 - 3x - 1) = x^2 - 4x + 8 - 2x^2 + 3x + 1 = -x^2 - x + 9.$$

Multiplying two polynomials: we multiply each monomial of the first polynomial with each monomial of the second polynomial and simplify all those products to the resulting product polynomial

$$\begin{aligned}(2x^2 + 3y) \cdot (4x^2 - y) &= 2x^2(4x^2 - y) + 3y(4x^2 - y)\\ &= 2x^2 \cdot 4x^2 + 2x^2 \cdot (-y) + 3y \cdot 4x^2\\ &\quad + 3y \cdot (-y)\\ &= 8x^4 - \boxed{2x^2y} + \boxed{12x^2y} - 3y^2\\ &= 8x^4 + 10x^2y - 3y^2.\end{aligned}$$

1.2 Equations in one variable

Anticipating this paragraph we refresh some vocabulary for it. A **solution** is any value assigned to the variable that turns the given equation into an *equality* (being *true*). The **scope** of an equation is any number set in which the equation resides, realizing it will be most likely $\mathbb{R}$. We define the **solution set** as the set containing all legal solutions to an equation. This solution set always is a subset of the scope of the equation.

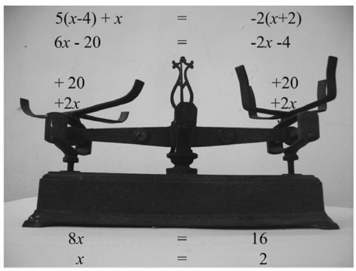
Linear Equations

A **linear equation** is an algebraic equation of degree one, referring to the maximum natural exponent of the unknown quantity. By simplifying we can always standardize any linear equation to

$$ax + b = 0, \tag{1.1}$$

given $a \in \mathbb{R} \setminus \{0\}$ and $b \in \mathbb{R}$. We cite $3x+7=22, 5x-9d=c$ and $5(x-4)+x=-2(x+2)$ as examples of linear equations, and $3x^2+7=22$ and $5ab-9b=c$ as counterexamples. The adjective 'linear' originates from the Latin word 'linea' meaning (straight) line as referring to the graph of a linear function (see chapter 4).

We solve a linear equation for its unknown part by rewriting the entire equation until its shape exposes the solution explicitly.



We recall easily the required rules for rewriting a linear equation by the metaphor denoting a linear equation as a 'pair of scales'. This way we should never forget to keep the equation's balance: whatever operation we apply, it has to act on both sides of the equals-sign. If we add (or subtract) to the left hand 'scale' than we are obliged to add (or subtract) the same term to the right hand 'scale'. If we multiply (or divide) the left hand side, than we are likewise obliged to multiply (or divide) the right hand side with the same factor. If not, our equation would loose its balance just like a pair of scales would. We realize that our metaphor covers all usual 'rules' to handle linear equations.

The reason we perform certain rewrite steps depends on which variable we are aiming for. This is called *strategy*. Solving the equation for a different variable implies a different sequence of rewrite steps.

Example: We solve the equation $5(x-4)+x=-2(x+2)$ for x. Firstly, we apply the distributive law: $5x-20+x=-2x-4$. Secondly, we put all terms dependent of x to the left hand side and the constant numbers to the right hand side $5x+x+2x=-4+20$. Thirdly, we simplify both sides $8x=16$. Finally, we find $x=2$ leading to the solution singleton $\{2\}$.

Quadratic Equations

Handling quadratic expressions and solving quadratic equations are useful basics in order to study topics in multimedia, digital art and technology.

▷ Expanding products

We refresh **expanding** a product as (repeatedly) applying the distributive law until the initial expression ends up as a pure *sum* of terms. Note that our given polynomial $V(x)$ itself does not change: we just shift its appearance to a pure sum. We illustrate this concept through $V(x)=(2x-3)(4-x)$.

$$\begin{aligned}(2x-3)(4-x) &= (2x-3)\cdot 4+(2x-3)\cdot(-x)\\ &= (8x-12)+(-2x^2+3x)\\ &= -2x^2+11x?12.\end{aligned}$$

Other examples are

$$5a(2a^2-3b)=5a\cdot 2a^2-5a\cdot 3b=10a^3-15ab$$

and

$$\begin{aligned}4\left(x-\frac{1}{2}\right)\left(x+\frac{13}{2}\right) &= (4x-2)\left(x+\frac{13}{2}\right)\\ &= (4x-2)\cdot x+(4x-2)\cdot\frac{13}{2}\\ &= 4x^2-2x+26x-13=4x^2+24x-13.\end{aligned}$$

▷ Factoring polynomials

We define **factoring** a polynomial as decomposing it into a pure *product* of (as many as possible) factors. Note that our given polynomial $V(x)$ itself does not change: we just shift its appearance to a pure product. Our **trinomial** $V(x)=-2x^2+11x-12$ just shifts its appearance to the pure product $V(x)=(2x-3)(4-x)$ when factored. It merely shows that the product $(2x-3)(4-x)$ is a factorization of the trinomial $-2x^2+11x-12$.