Numerical Methods for Ordinary Differential Equations

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Preface

In this book we discuss several numerical methods for solving ordinary differential equations. We emphasize the aspects that play an important role in practical problems. We confine ourselves to ordinary differential equations with the exception of the last chapter in which we discuss the heat equation, a parabolic partial differential equation. The techniques discussed in the introductory chapters, for instance interpolation, numerical quadrature and the solution to nonlinear equations, may also be used outside the context of differential equations. They have been included to make the book self-contained as far as the numerical aspects are concerned. Chapters, sections and exercises marked with a * are not part of the Delft Institutional Package.

The numerical examples in this book were implemented in Matlab, but also Python or any other programming language could be used. A list of references to background knowledge and related literature can be found at the end of this book. Extra information about this course can be found at `http://NMODE.ewi.tudelft.nl`, among which old exams, answers to the exercises, and a link to an online education platform. We thank Matthias Möller for his thorough reading of the draft of this book and his helpful suggestions.

Delft, January 2015
C. Vuik

The figure at the cover shows the Erasmus bridge in Rotterdam. Shortly after the bridge became operational, severe instabilities occurred due to wind and rain effects. In this book we study, among other things, numerical instabilities and we will mention bridges in the corresponding examples. Furthermore, numerical analysis can be seen as a bridge between differential equations and simulations on a computer.

Contents

Chapter 1

Introduction

1.1 Some historical remarks

Modern applied mathematics started in the 17th and 18th century with scholars like Stevin, Descartes, Newton and Euler. Numerical aspects found a natural place in the analysis but the expression "numerical mathematics" did not exist at that time. However, numerical methods invented by Newton, Euler and at a later stage by Gauss still play an important role even today.

In the 17th and the 18th century fundamental laws were formulated for various subdomains of physics, like mechanics and hydrodynamics. These laws took the form of simple looking mathematical equations. To the disappointment of many scientists, these equations could be solved analytically in a few special cases only. For that reason technological development has only been loosely connected with mathematics. The introduction and availability of digital computers has changed this. Using a computer it is possible to gain quantitative information with detailed and realistic mathematical models and numerical methods for a multitude of phenomena and processes in physics and technology. Application of computers and numerical methods has become ubiquitous. Statistical analysis shows that non-trivial mathematical models and methods are used in 70% of the papers appearing in the professional journals of engineering sciences.

Computations are often cheaper than experiments; experiments can be expensive, dangerous or downright impossible. Real life experiments can often be performed on a small scale only, which makes their results less reliable.

1.2 What is numerical mathematics?

Numerical mathematics is a collection of methods to approximate solutions to mathematical equations numerically by means of *finite* computational processes.

In large parts of mathematics the most important concepts are mappings and sets. In numerical mathematics the concept of computability should be added. Computability means that the result can be obtained in a finite number of operations (so the computation time will be finite) on a finite subset of the rational numbers (because a computer has only finite memory).

In general the result will be an approximation of the solution to the mathematical problem, since most mathematical equations contain operators based on infinite processes, like integrals and derivatives. Moreover, solutions are functions whose domain and image may (and usually do) contain irrational numbers.

Because, in general, numerical methods can only obtain approximate solutions, it makes sense to apply them only to problems that are insensitive to small perturbations, in other words to problems that are *stable*. The concept of stability belongs to both numerical and classical mathematics. An important instrument in studying stability is functional analysis. This discipline

Chapter 2

Interpolation

2.1 Introduction

In practical applications it is frequently the case that only a limited amount of measurement data is available, from which intermediate values should be determined (interpolation) or values outside the range of measurements should be predicted (extrapolation). An example is the number of cars in The Netherlands, which is tabulated in Table 2.1 (per 1000 citizens) from 1990 every fifth year up to 2010. How can these numbers be used to estimate the number of cars in intermediate years, e.g. in 2008, or to predict the number in the year 2020?

Table 2.1: Number of cars (per 1000 citizens) in The Netherlands (source: Centraal Bureau voor de Statistiek).

year	1990	1995	2000	2005	2010
number	344	362	400	429	460

In this chapter several interpolation and extrapolation methods will be considered.

A second example of the use of interpolation techniques is image visualization. By only storing a limited number of pixels, much memory can be saved. In that case, an interpolation curve should be constructed in order to render a more or less realistic image on the screen.

A final application is the computation of trigonometric function values on a computer, which is time consuming. However, if a number of precalculated function values is stored in memory, the values at intermediate points can be determined from these in a cheap way.

2.2 Linear interpolation

The simplest way to interpolate is zeroth degree (constant) interpolation. Suppose the function value at a certain point is known. Then, an approximation in the neighborhood of this point is set equal to this known value. A well known example is the prediction that tomorrow's weather will be the same as today's. This prediction appears to be correct in 80% of all cases (and in the Sahara this percentage is even higher).

A better way of interpolation is a straight line between two points (see Figure 2.1). Suppose that the function values $f(x_0)$ and $f(x_1)$ at the points x_0 and x_1 are known ($x_0 < x_1$). In that case, it seems plausible to take the linear function (a straight line) through $(x_0, f(x_0))$, $(x_1, f(x_1))$, in

Chapter 3

Numerical differentiation

3.1 Introduction

Everyone who possesses a car and/or a driver's licence is familiar with speeding tickets. In The Netherlands, speeding tickets are usually processed in a fully automated fashion, and the perpetrator will receive the tickets within a couple of weeks after the offence. The Dutch police optimized the procedures of speed control such that this effort has become very profitable to the Dutch government. Various strategies for speed control are carried out by police forces, which are all based on the position of the vehicle at consecutive times. The actual velocity follows from the first-order derivative of the position of the vehicle with respect to time. Since no explicit formula for this position is available, the velocity can only be estimated using an approximation of the velocity based on several discrete vehicle positions at discrete times. This motivates the use of approximate derivatives, also called *numerical derivatives*. If the police want to know whether the offender drove faster before speed detection (in other words, whether the perpetrator hit the brakes after having seen the police patrol), or whether the driver was already accelerating, then they are also interested in the acceleration of the 'bad guy'. This acceleration can be estimated using numerical approximations of the second-order derivative of the car position with respect to time.

Since the time-interval of recording is nonzero, the velocity is not determined exactly in general. In this chapter, the resulting error, referred to as the *truncation error*, is estimated using Taylor series. In most cases, the truncation error increases with an increasing size of the recording interval (Sections 3.2 and 3.4). Next to the truncation error, the measurement of the position of the vehicle is also prone to measurement errors. Issues that influence the results are, for example, parallax, the measurement equipment, and in some cases even the performance of the police officer (in car-videoing and laser control). These measurement errors provide an additional deterioration of the approximation of the speed and acceleration. The impact of measurement errors on approximations of derivatives is treated in Section 3.3.

3.2 Simple difference formulae for the first derivative

Suppose f is a continuously differentiable function. The *forward difference* is defined as

$$Q_f(h) = \frac{f(x+h) - f(x)}{h}, \quad h > 0,$$

in which h is called the *step size*. By definition,

$$\lim_{h \to 0} \frac{f(x+h) - f(x)}{h} = f'(x),$$

Chapter 7

The finite-difference method for boundary-value problems

7.1 Introduction

Many applications can be simulated by solving a boundary-value problem. A one-dimensional boundary-value problem consists of a differential equation on a line segment, where the function and/or its derivatives are given at *both* boundary points.

Stationary heat conduction in a bar
As an example we consider the temperature distribution in a bar with length L and cross-sectional area A (Figure 7.1).

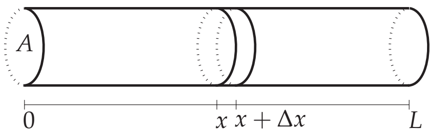

Figure 7.1: Bar with length L and cross-sectional area A.

We denote the temperature in the bar by $T(x)$ (measured in K), and assume that the temperature is known at both ends: $T(0) = T_l$ and $T(L) = T_r$. Further, heat is assumed to be generated inside the bar. We denote this heat production by $Q(x)$ $(J/(m^3s))$. In general, the temperature profile evolves over time towards a steady state. In this example we are interested in the temperature after a long period of time, i.e. the steady-state temperature. For the derivation of the differential equation the energy conservation law is applied to the control volume between x and $x + \Delta x$ (see Figure 7.1).

There is heat transport by conduction through the faces in x and $x + \Delta x$. According to Fourier's law this heat transport per unit area and per unit time is called the *heat flow density*, and equals

$$q(x) = -\lambda \frac{dT}{dx}(x),$$

where λ $(J/(msK))$ is called the *heat-conduction coefficient*. For the control volume between x and $x + \Delta x$, the energy balance should be satisfied: the total heat outflow at $x + \Delta x$ minus the total heat inflow at x should equal the amount of heat produced in this segment. This can be expressed as

$$-\lambda A \frac{dT}{dx}(x + \Delta x) + \lambda A \frac{dT}{dx}(x) = AQ(x)\Delta x.$$